VALLIAMMAI ENGINEERING COLLEGE
SRM Nagar, Kattankulathur-603203.

DEPARTMENT OF MATHEMATICS
QUESTION BANK
I-M.E(CIVIL) MA7154-ADVANCED MATHEMATICAL METHODS

UNIT-I : Laplace Transform Techniques For Partial Diff. Equations

PART-A

1.State the existence of Laplace Transform

2. Find the Laplace transform of t^n ,where n is a positive integer

3. *If* $L(f(t) = F(s)$, then prove that $L[e^{at} f(t)] = F(s-a)$

4. Find $L[t\sin^2 t]$, 5. Find $L[t^2 \cos 5t)$, 6. Find $L[t\,e^{-4t} \sin 3t]$

7. State Initial and Final value theorems for Laplace transform

8. Find $L[\frac{1-e^t}{t}]$

9. State and prove that change of scale property in Laplace transforms

10. Define Laplace transform of Error function

11. Find $L^{-1}(\frac{1}{s+1})^2$

12. Find the Laplace transform of $(f(t)/t)$

13. Find $L^{-1}[\frac{s+2}{s^2-4s+13}]$ 14. Find $L^{-1}[\frac{1}{s(s+1)(s+2)}]$

15. Find the inverse Laplace transform of $\log[\frac{s+a}{s+b}]$

16. Define Laplace transform of unit step function and find its Laplace transform

17. If L[f(t)]= F(s), then prove that $L[\int_0^t f(t)\,dt] = \frac{F(s)}{s}$

18.Solve the integral equation $y' + 3y + 2\int_0^t y dt = t$, given that y(0)=0

19.Using Laplace transform method, solve $\frac{dy}{dt} - y = 2$,given that y(0)=2

20.If u(x,t) is a function of two variables x and t, prove that $L[\frac{\partial u}{\partial t}; s] = sU(x,s) - u(x,0)$

PART-B

1. Find (i) $L[t\sin 5t\cos 2t]$ (ii) $L[\sin^3 t]$ $(iii)\,L[t\,e^t \sin 2t]$ $(iv)L[\frac{\cos 2t - \sin 3t}{t}]$

2. (i) Verify Initial Value Theorem and Final Value Theorem for $f(t) = 1 + e^{-t}(\sin t + \cos t)$

   (ii)Find the Laplace Transform of $erf(t^{\frac{1}{2}})$

3. (i) Derive the Laplace transform of error function and use it, find $L(erf(\sqrt{t})$

   (ii)Find the Laplace Transform of $f(t) = \begin{cases} \frac{t}{b}, & for\ 0 < t < a \\ \frac{2b-t}{b}, & for\ a < t2a \end{cases}$ given that f(t+2a)=f(t) for t>0

4 . (i) State and prove Convolution theorem in Laplace trans forms,

   (ii) Using Convolution theorem, find the inverse Laplace Transform of $\frac{s}{(s^2+9)(s^2+25)}$

5. (i) Find the Laplace transform of (i) $J_0(t)$ (ii) $t\,J_0(t)$

   (ii) Apply the convolution theorem to evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2};t\right]$

6. (i) Solve the simultaneous equations $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$,using the Laplace transform method which satisfies the conditions x(0)=1,y(0)=0

   (ii) Find the solution of the initial value problem $y'' - 4y' + 4y = 12t^2 e^{-2t}$, $y(0) = 2$ and $y'(0) = 1$

7. Using Laplace transform, solve the IBVP $PDE: u_t = 3u_{xx}$, $BCs: u(\frac{\pi}{2},t) = 0,\ u_x(0,t) = 0,$

   $ICs:\ u(x,0) = 30\cos 5x.$

8. Using Laplace transform, solve the IBVP

   $u_{tt} = u_{xx}\ \ 0 < x < l, t > 0,\ u(x,0) = 0\ , u_t(x,0) = \sin(\frac{\pi x}{l})\ , 0 < x < l,$

   $u(0,t) = 0\ and\ u(l,t) = 0, t > 0$

9. Using Laplace transform, solve the IBVP

   $PDE: u_{tt} = u_{xx}\ \ 0 < x < 1, t > 0,\quad BCs: u(x,0) = u(1,t) = 0, t > 0$

   $ICs:\ u(x,0) = \sin \pi x,\ u_t(x,0) = -\sin \pi x,\ 0 < x < 1$

10. A string is stretched and fixed between two fixed points (0,0) and (l,0). Motion is initiated by displacing the string in the form $u = \lambda \sin(\frac{\pi x}{l})$ and released from rest at time t=0.Find the displacement of any point on the string at any time t.

UNIT-II : Fourier Transform Techniques For Partial Diff. Equations

PART-A

1. Define Fourier transform
2. State Fourier Integral theorem
3. Write the existence conditions for the Fourier transform of the function(x)
4. State Dirichlet's conditions
5. Give Fourier transform pairs
6. Find the Fourier Transform of $f(x) = \begin{cases} 1, in\ |x| < a \\ 0, in\ |x| > a \end{cases}$
7. State and prove modulation property in Fourier transform
8. State and prove change of scale property in Fourier transform
9. State and prove shifting property in Fourier transform
10. Find the Fourier Sine transform of e^{-2x}
11. Find the Fourier Sine transform of 1/x
12. Find the Fourier Sine transform of $e^{-|x|}$
13. Find the Fourier Sine transform of e^{-at^2}
14. Find the Fourier Cosine transform of f(x)=$\frac{1}{1+x^2}$
15. If F(α) is the Fourier Transform of f(x), then the Fourier Transform of f(ax) is $\frac{1}{|a|}F(\alpha/a)$
16. Define Dirac delta function on Fourier transform.
17. State convolution Theorem on Fourier Transform
18. State Parseval's identity on Fourier Transform
19. Using Transform method, evaluate $\int_0^\infty \frac{dx}{(x^2+9)^2}$
20. Using Transform method, evaluate $\int_0^\infty \frac{x^2dx}{(x^2+1)^2}$

PART-B

1 (i) Find the Fourier Transform of $f(x) = e^{\frac{-x^2}{2}}$

(ii)Find the Fourier Transform of $\begin{cases} x, in\ 0 < x < 1 \\ 2-x, in\ 1 < x < 2 \\ 0,\ in\ x > 2 \end{cases}$

2. Find the Fourier Cosine and Sine Transform of e^{-2x} and evaluate the integrals

(i) $\int_0^\infty \frac{\cos\alpha x}{\alpha^2+4}d\alpha$ (ii) $\int_0^\infty \frac{\alpha\sin\alpha x}{\alpha^2+4}d\alpha$

3. (i) State and prove Parseval's identity the Fourier transforms

(ii) Find the Fourier Transform of $f(x) = \begin{cases} 1, & if \ |x| \le a \\ 0, & else \end{cases}$ hence deduce that $\int_0^\infty (\frac{\sin t}{t}) dt = \frac{\pi}{2}$

4. (i)If the Fourier sine transform of f(x) is $\frac{\alpha}{1+\alpha^2}$, find f(x).

(ii)Find the Fourier Transform of $f(x) = \begin{cases} 1-|x|, & if \ |x| < 1 \\ 0, & if \ |x| > 1 \end{cases}$ hence deduce that $\int_0^\infty (\frac{\sin t}{t})^4 dt = \frac{\pi}{3}$

5. (i) Find the Fourier Transform of f(x) defined by $f(x) = \begin{cases} 1-x^2, & if \ |x| < 1 \\ 0, & if \ |x| > 1 \end{cases}$ hence evaluate

$$\int_0^\infty (\frac{x\cos x - \sin x}{x^3}) \cos\frac{x}{2} dx.$$

6. (i) Find the Fourier Transform of $f(x) = \begin{cases} 1, & if \ |x| < a \\ 0, & if \ |x| > a \end{cases}$ hence deduce that

$$\int_0^\infty (\frac{\sin t}{t}) dt = \frac{\pi}{2} \ and \int_0^\infty (\frac{\sin t}{t})^2 dt = \frac{\pi}{2}$$

7. (i) Using Fourier Sine Transform, prove that $\int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+16)} = \frac{\pi}{14}$

(ii) Find the Fourier transform of $e^{-a|x|} \ if \ a > 0$, deduce that $\int_0^\infty \frac{dx}{(x^2+a^2)} = \frac{\pi}{4a^3}, if \ a > 0$

8. A one-dimensional infinite solid, $-\infty < x < \infty$,is initially at temperature F(x). For times t>0, heat is generated within the solid at a rate of g(x ,t) units, Determines the temperature in the solid for t>o. (OR)

Using the finite Fourier transform, solve the BVP described by PDE:
$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, 0 < x < 6, t > 0$, subject to BC: $V_x(0,t) = 0 = V_x(6,t)$, IC :V(x,0)=2x.

9. A uniform string of length L is stretched tightly between two fixed points at x=0 and x=L. If it is displaced a small distance $\in$ at a point x=b, 0<b<L, and released from rest at time t=0, find an expression for the displacement at subsequent times.

10.Using the method of integral transform, solve the following potential problem in the semi-infinite Strip described by $PDE : u_{xx} + u_{yy} = 0 \ , 0 < x < \infty \ , 0 < y < a$

Subject to BCs: u(x,0)=f(x) , u(x,a)=0, u(x,y)=0 ,

$0 < y < a \ , 0 < x < \infty \ and \ \frac{\partial u}{\partial x} \ tend \ to \ zero \ as \ x \to \infty$

UNIT-III : Calculus of Variations

PART -A

1. Define a functional
2. Write Euler's equation for functional.
3. Define isoperimetric problems.
4. Define several dependent variables.
5. Write a formula for functional involving higher order derivatives.
6. If y is independent of y, then give the reduced form of the Euler's equation
7. Define ring method.
8. State Brachistochrone problem.
9. Write other forms of Euler's equation.
10. Write the ostrogradsky equation for the functional $I[z(x,y)] = \iint_D \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + zf(x,y)\right\} dxdy$
11. Show that the symbols $\frac{d}{dx} and\ \ \delta\ commutative$
12. Write Euler-Poisson equation.
13. Define moving boundaries.
14. Find the transversality condition for the functional $v = \int_{x_0}^{x_1} A(x,y).\sqrt{\left[1+(\frac{dy}{dx})^2\right]}dx$
15. Test for an extremum the functional $I[y(x)\} = \int_0^1 (xy + y^2 - 2y^2 y')\,dx$ y(0)=1, y(1)=2.
16. Write down biharmonic equation.
17. State Hamilton's Principle
18. State du Bois-Reymond
19. Explain direct methods in variational problem
20. Explain Rayleigh-Ritz method.

PART –B

1. (i) Find the extremals of $\int_{x_0}^{x_1}(y^2 + y^{1^2} - 2ye^x)\,dx$

   (ii) Find the extremals of $\int_{x_0}^{x_1}(y^{1^2} + 2yy^1 \;\; - 16y^2)\,dx$

2. (i)Solve the extremal $v[y(x)] = \int_{x_0}^{x_1} \frac{dy}{dx}\left(1 + x^2 \frac{dy}{dx}\right) dx.$

   (ii)Find the extremal of the function$\int_{x_0}^{x_1} \frac{y^{1^2}}{x^3}\,dx$

3. (i)Show that the straight line is the sharpest distance between two points in a plane.

   (ii) A curve c joining the points (x_1, y_1) and (x_2, y_2) is revolved about the x-axis. Find the shape of the curve, so that the surface area generated is a minimum.

4. (i)State and Prove Brachistochrone problem.(ii) Show that the curve which extremize $I = \int_0^{\frac{\pi}{2}}[(y^{11})^2 - y^2 + x^2]dx$ given that $y(0) = 0, y^1(0) = 1, y\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, y^1\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\ is\ y = \sin x.$

5. Determine the extremal of the functional $I[y(x)] = \int_{-a}^{a}\left\{\frac{1}{2}\mu(y^{11})^2 + \rho y\right\}dx$ that satisfies the boundary condition $y(-a) = 0, y(a) = 0, y^1(-a) = 0, y^1(a) = 0.$

6. (i)Find the plane curve of a fixed perimeter and maximum area.

   (ii)Prove that the sphere is the solid figure of a revolution which for a given surface has maximum volume.

7. Find the curve on which an extremum of the function $I = \int_0^{\frac{\pi}{4}}\left\{y^2 - \left(\frac{dy}{dx}\right)^2\right\}dx, y(0) = 0$ can be achieved if the second boundary point is permitted to move along the straight line $x = \frac{\pi}{4}$.

8. Solve the boundary value problem
   $y^{11} - y + x = 0\ (0 \le x \le 1)\ y(0) = y(1) = 0$ by Rayleigh Ritz method.

9. (i)Find the plan curve of fixed perimeter and maximum area.(ii) To determine the shape of a solid of revolution moving in a flow of gas with least resistance.

10. Derive the Euler's equation and use it, find the extremals of the functional

$$V[y(x)\} = \int_{x_0}^{x_1}(y^2 + y'^2 - 2y\sin x)\,dx.$$

UNIT-IV : Conformal Mapping and its Applications of Variations

PART-A

1. Define Conformal Mapping and give an example.
2. Define Isogonal trandformation
3. State the condition for the function f(z) to be conformal
4. Define Mobius transformation
5. Define cross ratio of $Z_1 \, , Z_2 \, , Z_3 \, , Z_4$
6. Define Schwarz-Christoffel transformation.
7. Define fixed and invariant point of the transformation w=f(z)
8. Find the fixed points of the transformation $w = \dfrac{2z-5}{z+4}$
9. Find the critical point of the transformation $W^2 = (Z-\alpha)(Z-\beta)$
10. Show that by means of the transformation W=1/Z the circle C given by $|Z-3| = 5$ is mapped into the circle $|Z+3/16| = 5/16$
11. Find the Jacobian of the transformation $W = 2Z^2 - iZ + 3 - i$
12. Show that the transformation $W = 2Z - 3i\overline{Z} + 5 - 4i$ is equivalent to u=2x+3y+5, v=2y-3x-4
13. Define harmonic function.
14. Show that the function sinxcoshy is harmonic
15. Define Heat flux of heat flow.
16. State the basic assumptions of fluid flow.
17. Define streamline
18. Define velocity potential, source and sink
19. Define complex temperature of heat flow
20. Discuss the motion of a fluid having complex potential $\Omega(z) = ik \log z \;\; where \;\; k > 0$

PART-B

1. (i) Prove that the bilinear transformation transforms circles of the z plane into circles of the w plane, where by circles we include circles of infinite radius which are straight lines
   (ii) Define Bilinear transformation and find the bilinear transformation which maps $z_1 = -2 \, , z_2 = 0 \, , z_3 = 2 \; on \; to \; the \; points \; w_1 = \infty \, , w_2 = 1/2 \, , w_3 = 3/4.$
2. (i) Prove that $\dfrac{\partial(u,v)}{\partial(x,y)} \dfrac{\partial(x,y)}{\partial(u,v)} = 1$
   (ii) Discuss the Schwarz- Christoffel transformation
3. (i) Define cross ratio of z_1, z_2, z_3 and Critical points.
   (ii)Find a BLT $z_1 = 0 \, , \; z_2 = -i \; , z = -1 \;\; in \; to \; the \;\; points \;\; w_1 = 1, \;\; w_2 = 1 \;\; w_3 = 0$

4. (i) Find a transformation which maps a polygon in the w plane on to the unit circle in the ζ plane
   (ii) Find a function which maps the interior of a triangle in the w plane on to the upper half of the z plane.
5. (i) Prove that a harmonic function $\Phi(x,y)$ remains harmonic under the transformation w=f(z) where f(z) is analytic and $f'(z) \neq 0$
   (ii) Find a function harmonic in the upper half of the z plane, Im{z}>0, which takes the prescribed values on the x-axis given by $G(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$
6. (i) Define Harmonic function and show that the function $u = \log(x^2 + y^2)$ is harmonic.
   (ii) Prove that $\dfrac{\partial^2\Phi}{\partial x^2} + \dfrac{\partial^2\Phi}{\partial y^2} = |f'(z)|^2\left(\dfrac{\partial^2\Phi}{\partial u^2} + \dfrac{\partial^2\Phi}{\partial v^2}\right)$ where w=f(z) is analytic and $f'(z) \neq 0$
7. Find the complex potential due to a source at z=-a and a sink at z=a equal strengths k, Determine the equipotential lines and streamlines and represent graphically and find the speed of the fluid at any point.
8. (i)Find the complex potential for a fluid moving with constant speed V_0 in a direction making an angle δ with the positive x axis
   (ii) Determine the velocity potential and stream function
   (iii) Determine the equations for the streamlines and equipotential lines.

9. The complex potential of a fluid flow is given by $\Omega(z) = V_0(z + \dfrac{a^2}{z})$ where V_0 and α are positive constants.(a) Obtain equations for the streamlines and equipotential lines, represent them graphically and interpret physically.(b) Show that we can interpret the flow as that around a circular obstacle of radius a.(c) Find the velocity at any point and determine its value far from the obstacle (d) Find the stagnation points.

10. Fluid emanates at a constant rate from an infinite line source perpendicular to the z plane at z=0 (a) Show that the speed of the fluid at a distance r from the source is V=k/r, where k is a constant(b) Show that the complex potential is $\Omega(z) = k \ln z$.(c) What modification should be made in (b) if the line source is at z=a? (d) What modification is made in (b) if the source is replaced by a sink in which fluid is disappearing at a constant rate?

UNIT-V : Tensor Analysis

PART-A

1. Define Tensor and Summation convention
2. Define Kronecker delta function of tensor
3. Define Covariant and Contravariant vectors
4. If A_i and B_i are covariant vector, show that $A_i B_i$ is covariant tensor of order 2
5. If A_i is a covariant and B^i is a contravariant vector, show that $A_i B^i$ is an invariant
6. Define $\delta^i{}_i$
7. Define reciprocal tensors
8. Define contraction of tensors
9. Define Symmetric and Skew-symmetric tensors
10. Define divergence of a contravariant vector
11. Prove that Kronecker delta (δ) is a mixed tensor of order 2.
12. Define Metric tensor and Conjugate tensor
13. State Christoffel symbols
14. If ϕ is a function of the n quantities x^i, write the differential of ϕ using the summation convention
15. Define divergence and curl of tensor
16. If $(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2$,find the value of [22.1] and [13,3]
17. Prove that $divA_i = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} A^k)}{\partial x^k}$.
18. prove that $\begin{Bmatrix} i \\ i \;\; j \end{Bmatrix} = \frac{\partial}{\partial x^i} (\log \sqrt{g})$
19. If A_{ij} is skew symmetric tensor, find the value of $(\left(\delta_j{}^i \;\; \delta_l{}^k + \delta_i{}^i \;\; \delta_j{}^k\right) A_{ik}$
20. Prove that $\frac{\partial g_{ij}}{\partial x^k} = [ik, j] + [jk, i]$

PART-B

1. (i)In a contravarant vector has components $\frac{dx}{dt}, \frac{dy}{dt}$ in rectangular coordinates, show that they are $\frac{dr}{dt}$ *and* $\frac{d\theta}{dt}$ in polar coordinates.

   (ii) If A^i is a contravariant vector and B_j is a covariant vector, prove that $A_i B^j$ is a mixed tensor of order 2.

(iii) If A^i is a contravariant vector, show that $\frac{\partial A^i}{\partial x^j}$ is not a tensor.

2. (i) State and prove Quotient law of tensor.

(ii) Find g *and* g^{ij} corresponding to the metric

$$(ds)^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3 \ .$$

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3. Find the components of the first and second fundamental tensor in spherical coordinates.

4. (i) If A^i is a contravariant vector, show that $\frac{\partial A^i}{\partial x^j}$ is not a tensor.

(ii)If A^i is an arbitrary contravariant vector and $C_{ij}A^iA^j$ is an invariant, show that $C_{ij} + C_{ji}$ is covariant tensor of order 2.

5 (i)Define Christoffel's symbols. Show that they are not tensors

(ii) Prove that the covariant derivative of g^{ij} is zero.

6. Given the covariant components in rectangular co-ordinates $2x - z,\ x^2y,\ yz$. Find the covariant components in (i) Spherical polar co-ordinates (r, θ, φ) (ii) Cylindrical co-ordinates(ρ, φ, z)

7. b) (i) Explain the covariant differentiation law for tensor.

(ii) Prove that $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^k}$.

8. Show that the tensors g_{ij}, g^{ij} and the Kroneckor delta δ_j^i are constants with respect to covariant differentiation.

9.(i) Find the g *and* g^{ij} for the metric

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1dx^2 + 4dx^2dx^3 + 0dx^3dx^1$$

(ii) Prove that the covariant derivative of g^{ij} is zero.

10. If $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2$, find the value of

(i) [22,1] and [13,3] (ii) $\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\}$ *and* $\left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\}$